Superstring representation of Hubbard model

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Abstract

The new method of investigation of strongly correlated electronic system is presented. Using this method we have derived the superstring from 2D Hubbard model. The novel expression for generalized supercoherent state has been calculated.

1. Nowadays the Hubbard model attracts considerable attention. This model is incorporated in several profound theoretical projects, proposed in [1] for example. These projects lead to deep interaction of ideas and methods of modern field theory and solid state physics [2]. In this article we present the new approach to the strongly correlating electronic systems. Using the functional integral we have derived the superstring from 2-D Hubbard model. Essential ingredient of this approach is the using of the supercoherent state, which exact expression was calculated.

2. Let us recall the formalism developed in [3] for systems with coulomb interactions.The system under consideration is described by Hubbard model:

$$H = -t \sum_{<\mathbf{r},\mathbf{r}'>,\mathbf{s}} {\alpha_{\mathbf{r},\mathbf{s}}}^{+} \alpha_{\mathbf{r}',\mathbf{s}} + U \sum_{\mathbf{r}} n_{\mathbf{r},\uparrow} n_{\mathbf{r},\downarrow} = \tag{1}$$

$$\sum_{\mathbf{r}} U\, {X_{\mathbf{r}}}^{22} + \sum_{A,C,\mathbf{r},\mathbf{r}'} t_{-AC}(\mathbf{r}-\mathbf{r}')\, {X_{\mathbf{r}}}^{-A}\, {X_{\mathbf{r}'}}^{C} \tag{2}$$

where $< r, r' >$-denote the sum over the nearest neighbors. Electrons are described by $(\alpha^+_{rs}, \alpha_{rs})$ and live in two dimensional plane $r = \{x, y\}$ (for example square lattice), s - spin of the electrons. We give two different forms of this model (the first form is a standard, the second contains the Hubbard operators X^A (in fact they are projectors $X^A_r = \mid pr >< qr \mid$). These operators

act in space of following states: $\mid 0>, \mid\uparrow>=\alpha_{\uparrow}^{+}\mid 0>, \mid\downarrow>=\alpha_{\downarrow}^{+}\mid 0>$ $, \mid 2>=\alpha_{\uparrow}^{+}\alpha_{\downarrow}^{+}\mid 0>$ (the space of eigenfunctions of Hubbard repulsion-the main and most complicated interaction in many interesting models), $A=(p,q); p,q=0,\uparrow,\downarrow,2$. X^{pq} has one nonzero element, sitting in p row and q coloum of 4×4 matrix. Local state of system and the Hilbert space we describe by supercoherent state:$\mid G[\chi(r,t'),E(r,t'),h(r,t')>$ where $\chi=\{\chi_1,\chi_2\}$ is dynamical two component odd valued grassmanian fields,$E=\{E_1,E_2,E_3\}$ and $h=\{h_1,h_2,h_3\}$ are three component even valued dynamical fields. χ describes electronic degree of fredoms, E and h parametrises charge density and spin density fluctuations appropreatly. According to this formulation the many particle systems are described by following effective functional:

$$L_{eff}=\frac{<G(\theta,\mathbf{r},t')\mid\left(\frac{\partial}{\partial t'}-H\right)\mid G(\theta,\mathbf{r},t')>}{<G(\theta,\mathbf{r},t')\mid G(\theta,\mathbf{r},t')>} \tag{3}$$

where $\mid G>$ -is supercoherent state, which is expressed through generators of the dynamic superalgebra, $\{\mathbf{r},t',\theta\}$ -supercoordinates of superspace. The classification of various types of (super)algebra in interactive systems was given in [4]. H is the Hubbard hamiltonian expressed through the infinite dimensional superalgebra generators $\{X_{\mathbf{r}}^{\alpha}\}$: where $U-$ on-site coulomb repulsion, $t_{\alpha,\beta}(\mathbf{r}-\mathbf{r'})$ - "interaction", arisen from kinetic energy. $\mid G>$ can be constructed by following way [3]:

$$\mid G>=e^{-\sum_{k=1}^{6}X^k\,b^k(\mathbf{r},t,\theta)-\sum_{j=1}^{2}X^{-j}\,\chi^j(\mathbf{r},t,\theta)}\mid 0>=\left(F\chi,z(E)+B\chi^2\right) \tag{4}$$

here $\mid 0>=\otimes_r\mid 0>_r$, $\{b^C\}=\{\{E_i\},\{h_i\}\}$ $i=1,2,3$ and $\{X^k\}=\{X^{02},X^{20},X^{00}-X^{22},X^{\uparrow\downarrow},X^{\downarrow\uparrow},X^{\uparrow\uparrow}-X^{\downarrow\downarrow}\}$,

$\{X^1,X^2\}=\{X^{0\uparrow}+X^{\downarrow 2},X^{0\downarrow}-X^{\uparrow 2}\}$- set of the bosonic fields (even valued grassmanian fields). We have following expansion for unity:$\int\mid G><G\mid d\mu(O)=1$ where $\mu(O)=<G\mid G>d^2\chi d^3Ed^3h$ is measure of all dynamical fields $O=\{O_s\}=\{\chi,E,h\}$ $\mid G>$ has four component, two of them are fermionic (odd valued grassmanian nonlinear composite fields), and two are bosonic (also composite and nonlinear in χ,E,h). Expanding the expression (4) in the form of infinite series and then summing some definite series that are matrix element of (4) we obtain following nonlinear expression:

$$F=a''+a'\,E_z+\mathbf{h}\,\sigma^{\mathbf{P}}\left(a'+a\,E_z\right),\quad B=\left(\{0,a\}+\rho(\psi)\,z(\tau)\,E^+\right) \tag{5}$$

where the prime means differentiation with respect to δ,σ_{μ}^{P} -Pauli matrix.

$$z(\tau)=\{\cos(\tau)+\cos(\theta')\,\sin(\tau),e^{i\,\phi'}\,\sin(\tau)\,\sin(\theta')\},\tau=\arcsin(\frac{E\,\psi}{\rho}),$$

$$\rho = \sqrt{E^2\,\psi^2 + \psi'^2}, \qquad \psi = \delta^5\,\frac{\partial\, f}{\partial\, E^2}, a = \frac{\partial^2 f}{E^2\,\partial\delta^2}$$

$$f = -h^{-2} + \frac{E^2\,\left(\frac{\sin(E\delta)}{E^3} - \frac{\sin(h\delta)}{h^3}\right)}{E^2 - h^2} \tag{6}$$

We use the spherical coordinate system $\{\, E, \theta', \phi' \,\}$ for $\mathbf{E}$. In (5,6) we put $\delta = 1$ after calculation.

3. The partition function of the system can be written as:

$$Z_{hub} = \int D\chi^*\, D\chi\, D\mathbf{E}\, D\mathbf{h}\, e^{-i\int d^2r\, dt'\, d^2\theta\, e^{\theta\,\theta^*}\, L(\mathbf{E},\mathbf{h},\chi)}$$

As usual [2],[3] ,we divide the time interval into N equal parts and finally take the limit:$N \rightarrow \infty$. Turning to three dimensional lattice we obtain following representation for:

$$e^{\int -id^2\mathbf{r}\, dt'\, L} = \prod_{n,m,k} e^{-i\, L(n,m,k)\Delta} = \prod_{n,m,k} (1 - i\, L(n,m,k)\Delta) \tag{7}$$

where $\{\, \mathbf{r}, t' \,\} = \{x, y, t'\} = \Delta\, \{n, m, k\,\}$. Now our intent is to show that some functional variables (for example $\{E^+, E^-\}$ can be exchanged by variables in space-time integral (x, y, t')). As a result of this transformation we will get some new functional representation for the partition function with different relations of the functional and the ordinary coordinates of integration. This representation will be the partition function of a superstring. Let us take the measure of fields component and represent this expression in following manner: $\int DE^+ = \prod_x \int dE_x^+$. Combining it with the expression (7) and expanding the infinite product to sum we can obtain the expression in first order of derivative:

$$\int \prod_x (1 - iL(x)\Delta)\; dE^+ = \sum_x \Delta\, \left(1 - i\int dE^+\, L(E^+(x))\right) =$$
$$\int DX\, e^{-i\int dE^+\, L(E^+)} \tag{8}$$

Now we consider all fields $\{\chi, \mathbf{h}, E_z\}$ as functions of $\{E^+, E^-\}$ only, for example: $E_z = E_z(\theta^{'}, \phi')$. In such way we get the reduction of the space-time variables.

4.Let us construct the map of two dimensional sphere parameterized by $\{E^+, E^-\}$ (we can as well select $\{h^+, h^-\}$, instead of $E^{\pm}$) to three dimensional space R^3. Now we exploit the results deduced in [5],[6] for the conformal immersion of two-dimensional surface that is string world sheet in R^3. We use the Gauss map in the same manner, as in [5]. This map is defined in our case by following procedure:

i) one make takes the two-dimensional sphere to be parameterized by spinor $z : (zz^* = 1)$. This Riemann sphere gives conformal compactification of two-dimensional plane and because of this it describes the superstring world sheet. Having the spinor z we can construct complex null three-dimensional vector:

$$t^+_\mu = \overline{z}\,\sigma_\mu z, \qquad t^-_\mu = (t^+_\mu)^\bullet \tag{9}$$

The important feature of null vector for us is that:

$${t_1}^2 + {t_2}^2 + {t_3}^2 = 0 \tag{10}$$

ii) For string, propagating in three-dimensional space, it is known that the ivolution equation takes the form:

$$\frac{\partial^2 X_\mu}{\partial\,\sigma^+\,\partial\,\sigma^-} = 0 \qquad (\partial_\pm X^\mu)^2 = 0 \tag{11}$$

It is obvious from (11) that real vectors $\frac{\partial X_\mu}{\partial \sigma^\pm}$ are null.

iii) From (9) we see that vector $\frac{\partial X_\mu}{\partial \sigma^\pm}$ and $t^\pm_\mu$ are tangent vectors to string world sheet. This similarities between (10) and (11) gives us the possibility to construct the mapping of R^2 to R^3 (Gauss map):

$$\partial_\pm X^\mu = \Omega\, t^\pm_\mu$$

Having obtained the Gauss map the question of reconstructing of the functional variables (coordinates of string world sheet) is answered by expression: $X_\mu(\sigma^+, \sigma^-) = \mathrm{Re}(\int^\sigma d\zeta\,\Omega(\zeta)\,t_\mu) + X_\mu(0,0)$ [5] ,[6] where Ω is given by equation:

$$\ln(\Omega)_\sigma = -\frac{t^\mu\,{t_\sigma}^\mu}{\mid t\mid^2}$$

Here the σ index means the derivative on σ. This equation is the integrability condition of the Gauss map.

5.Inserting ${t_\mu}^\pm$ into the following lagrangian:

$$2\int d^2\sigma\,{t_\mu}^-\,{t^+}_\mu = \int d^2\sigma \sqrt{(t^+ - t^-)^2\,(t^+ + t^-)^2} \;=$$

$$\int d^2\sigma \sqrt{\dot{X}^2 X'^2 - (\dot{X}\, X')^2} \tag{12}$$

we see that (12) is a Namby-Goto string [5]. For our purpose it is convenient to take this lagrangian in form given by Polyakov [7]:

$$L_{N-G} = \sqrt{g}\, g^{\alpha\beta} \frac{\partial X_\mu \partial X_\mu}{\partial \sigma^\alpha \partial \sigma^\beta} \tag{13}$$

where $g_{\alpha\beta}$ metric tensor of surface . Taking the $\widetilde{g}^{\alpha\beta} = \frac{\partial X_\mu \partial X_\mu}{\partial \sigma^\alpha \partial \sigma^\beta}$ we find that $g_{\alpha,\beta}$ can be expressed in following manner: $g_{\alpha,\beta} = \mathrm{C}\, \widetilde{g}_{\alpha,\beta}$ where C is arbitrary constant. It is seen that we deal with the conformal gauge for $\widetilde{g}$: $\widetilde{g}_{12} = \frac{-(z_1 z_2^* + z_2 z_1^*)^2}{2}, \widetilde{g}_{11} = \widetilde{g}_{22} = 0$. Now let us show how to derive the bosonic sector of superstring. Evaluating expression $(z_{\mathbf{r}}\ (\sum_{\mathbf{r}} U\, X^{pp}{}_{\mathbf{r}})\ z^*_{\mathbf{r}}\ )$ we get:

$$L_0 = \kappa_1 \left(1 - 2\, \sin(e)\, E^+ E^-\right) \left(1 + 2 \left(a'' - a'^2\, E_z{}^2\right)\right) \tag{14}$$

Chemical potential ϵ gives the additional term: $\kappa_2 npn$. Substituting expression of $t^\pm_\mu$ to (12) we obtain any function of $E_z(\theta', \phi)$:

By adding the lagrangian (13) to (12) we can obtain the string lagrangian if (14)is constrained by equation: $L_{N-G} + \sqrt{g} + L_0 = 0$

Returning to (2) we see that the Hubbard repulsion in continuum limit can be transformed to the string lagrangian:

$$L_{site} = \sqrt{g} + L_{N-G} \tag{15}$$

6. Effective functional can be evaluated as a series of a grassmanian fields:

$$L_{eff} = L_0 + L_2 + L_4 \tag{16}$$

Combining kinetic energy and (15) the following result can be obtained:

$$L = \sqrt{g}\, g^{\alpha\beta}\, \partial_\alpha X^\mu\, \partial_\beta X^\mu + \sqrt{g} + \Upsilon^{\alpha\beta}{}_{ij}\, \partial_\alpha \phi^i\, \partial_\beta \phi^j + \Psi^\bullet_e (e^\alpha \partial_\alpha + \Pi) \Psi_e + R_{ijkl} \chi^*_i \chi^*_j \chi_k \chi_l$$

where $\Psi_e = \{\chi, \chi^\bullet\}$ is two component spinor, and γ_i act in space of this components, $\Upsilon^{\alpha\beta}{}_{ij}\, \partial_\alpha \phi^i\, \partial_\beta \phi^j = (1 - npn)\nabla^2 A + \nabla^2 npn$ and $e^\alpha, \Pi, R_{ijkl}-$are given in first order of coefficients of superfield ,(the full expression for them are complex and will be given in other paper).

$$\begin{array}{ccc} e^\alpha & = (e)_1 + (e)_2 = & [C^{12}_{12}(1 - \sigma^1)\gamma^5 - (C^{12}_o \gamma^1 + c.c.)] + n\gamma^5 ... \\ \Pi & = & (t\nabla - \kappa_1)(e)_1 + (t\nabla - \kappa_2)(e)_2 + ... \\ R_{1221} & = & (t\nabla - \kappa_1) C^{12}_{12} - (t\nabla - \kappa_2) \det n + ... \end{array}$$

where $C^0_{12} = 2t(z_1B_2)^*det(F) + \kappa_1 z^*\gamma^5 B$, $C^{12}_{12} = 2t(det(F))^* z\gamma^1 B + \kappa_1 B^*\gamma^5 B + c.c$, $n^0_{12} = z^*B$, $n^{12}_{12} = B^*B$, $n = F^+F$, $npn = \sum_{i,j=1}^2 n_{ij}$, $A = z^*\gamma^5 z, \kappa_1 = U/2 - \epsilon, \kappa_2 = 2\epsilon - U/2$.

The operator ∇ equal to $\frac{\Delta}{\Omega}\sum_{i=\pm}\left[\sum_{\mu=1}^3\left(\frac{\partial X_\mu}{\partial\sigma_i}\right)^{-1}\right]\frac{\partial}{\partial\sigma_i}$ and acts on the right superfunction in any terms of the effective action: $\nabla(\Phi^*\Phi) = \Phi^*\nabla\Phi$. This condition follows from correct determination of continuum limit of kinetic energy.

References

[1] P.Anderson,Preprint of Princeton University, (1991),

Sher Alam, M.O.Rahman,T.Yanagisawa, H.Oyanagi, Quantum Group, String and HTSC material, Los Alamos archive: http://xxx.lanl.gov/archive/cond-mat/0110015

[2] E. Fradkin,Field Theories of condensed matter system, Addison-Wesley,(1991).

[3] V.Zharkov, Teor. Mat. Fiz.,**60**,404,(1984),**77**,107,(1988).

[4] V.Zharkov, Teor. Mat. Fiz.,**86**,262,(1991), **90**,75,(1992).

[5] P.Budinich,L.Dabrowski, Nuovo Cimento,**96**A,194,(1986).

[6] K. S. Viswanathan, R. Parthazarathy. Annals of Physics, **206**, 237, (1991).

[7] A. M.Polyakov,Gauge Field and Strings, Harwood Acad. Publ., (1987).